Error Statistics for Bias-Naïve Filtering in the Presence of Bias

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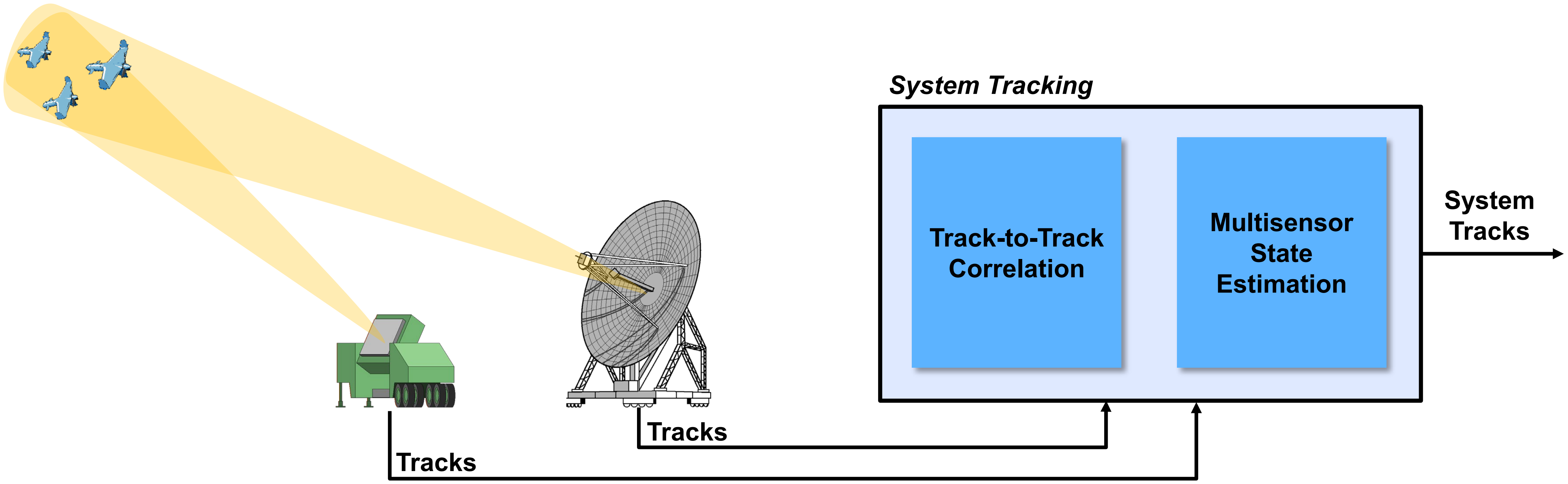
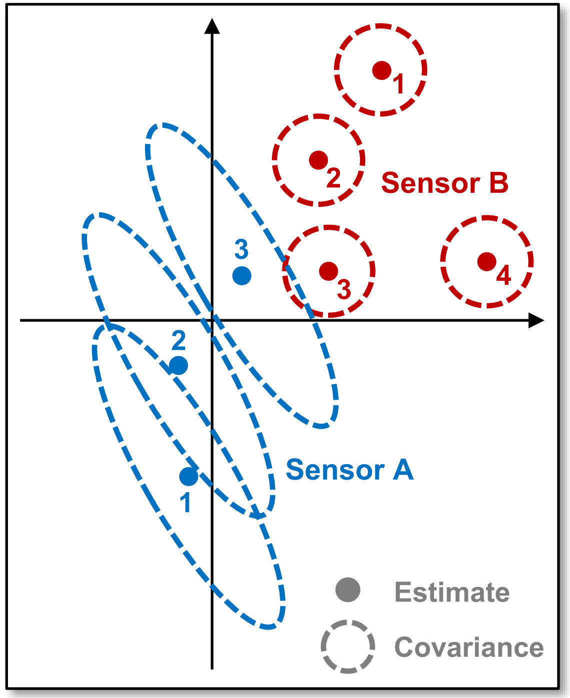
Multisensor, Multitarget Tracking

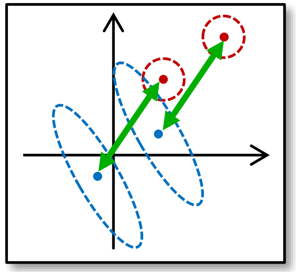


- **Many commercial and defense applications require the integration of tracking information from multiple sensors due to limitations such as coverage and varied phenomenology**
- **An unavoidable step in multisensor tracking is the association of tracks from one sensor to another, i.e., *track-to-track correlation***

Track-to-Track Correlation



Track-to-Track Correlation

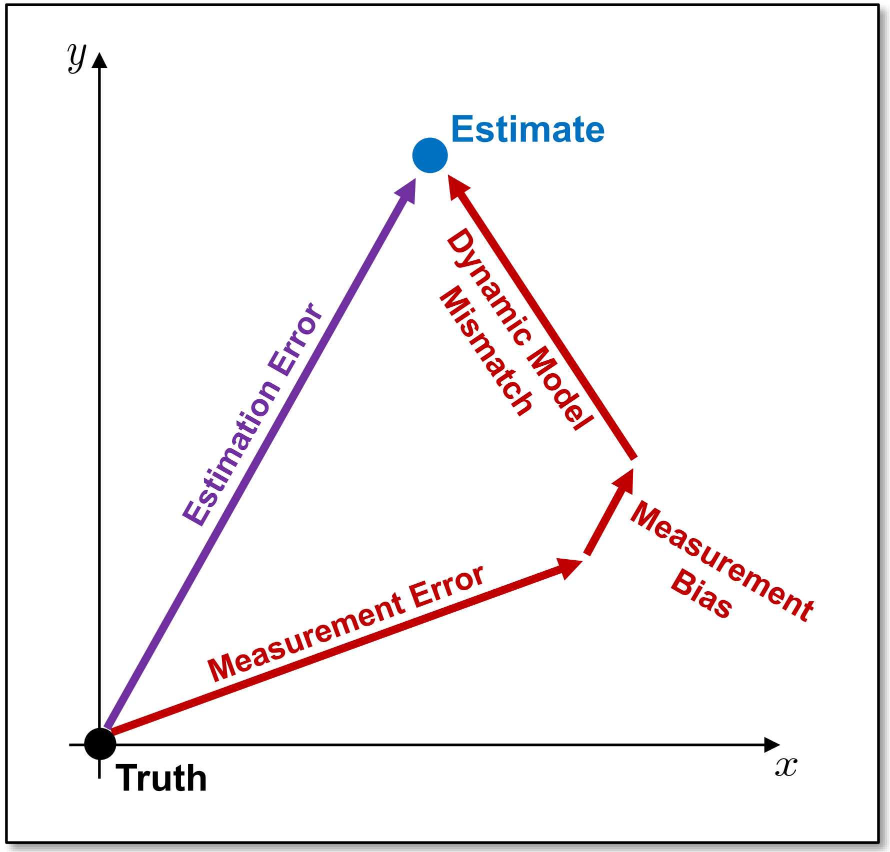


Translational Shift Assumption

- Tracks from multiple sensors need to be grouped as common targets before fusion of information
- Typical nuisance is measurement bias which differs from sensor to sensor
  - Commonly represented as a translational shift of the track states in position
  - Many track-to-track correlation algorithms hinge on this assumption
- In reality, estimation error from measurement bias can actually:
  - Vary from track to track within a single sensor
  - Alter positional derivatives, e.g., velocity, acceleration
- Important to understand to what degree this assumption is violated

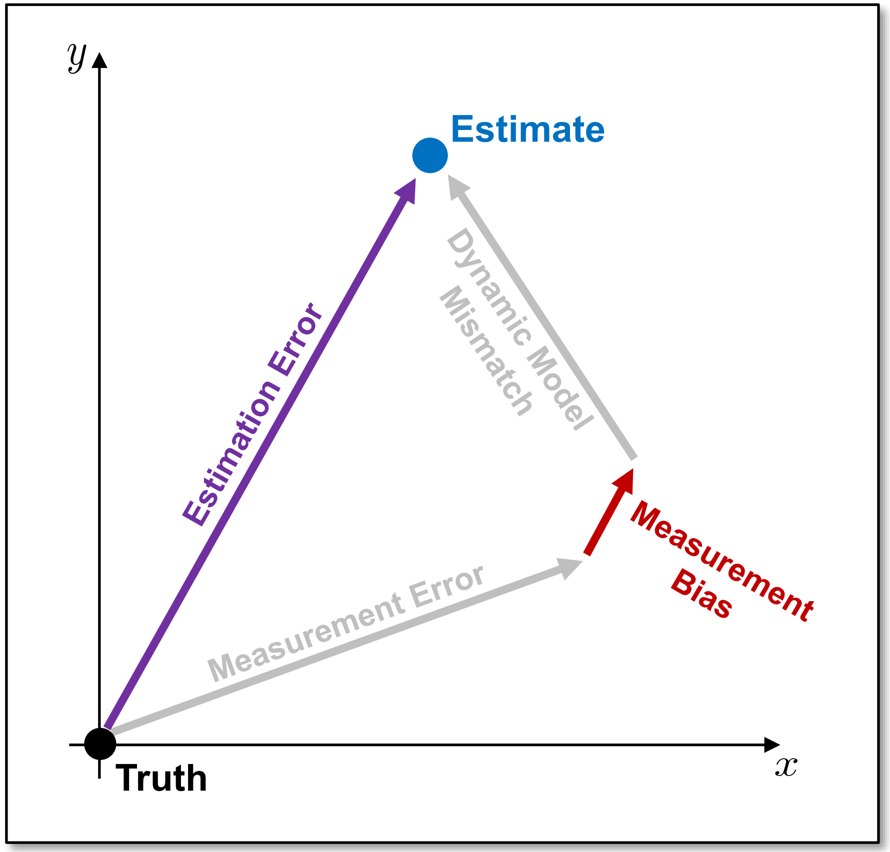
Estimation Error Contributions

- Error of an estimator is an aggregation of multiple factors:
  - Measurement error
  - Measurement bias
  - Dynamic model mismatch
  - …
- Depending on the system model, each factor can be a role of varying importance
- When designing a filter, it is critical to understand the sources of error and their relative significance in the total estimation error



Significance of Measurement Bias

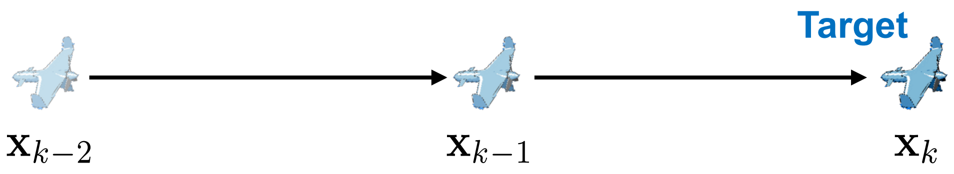
- **Objective is to quantify the effects of measurement bias on the error statistics of an estimator**
- **Related work:**
  - Performance analysis for reduced order filtering [Warren73], [Asher75]
  - Sensitivity analysis for model mismatch [Brown71], [Gelb74]
  - Covariance analysis with bias [Asher76], [Fitzgerald71]
- **This work has a narrower scope than previous treatments, allowing for:**
  - Analysis of linear and nonlinear systems
  - Consideration of deterministic and stochastic biases
  - Simpler, alternative derivations of related results



Outline

Linear Systems

Target Dynamics



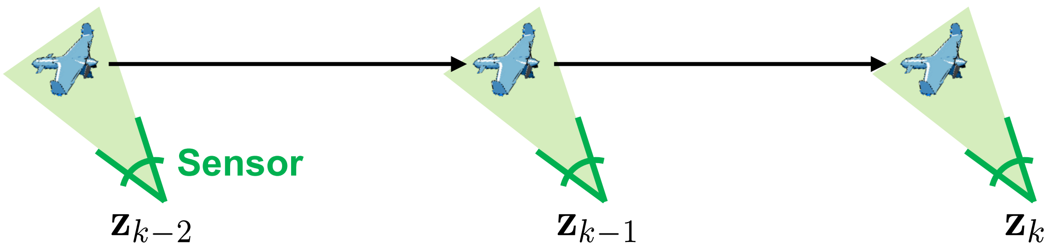
$$\mathbf{x}_k = \mathbf{\Phi}_{k,k-1}\mathbf{x}_{k-1} + \mathbf{v}_k$$

- Target state: $\mathbf{x}_k$
- State transition matrix: $\mathbf{\Phi}_{k,k-1}$
- Process noise[1]: $\mathbf{v}_k$

Target motion described with discrete-time, linear, recursive function with process noise

[1] $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$

Measurements



$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{b}_k + \mathbf{w}_k$$

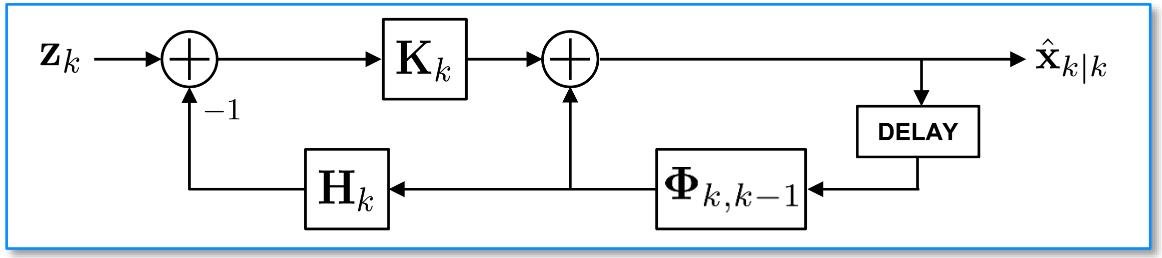
- Measurement: $\mathbf{z}_k$
- Measurement matrix: $\mathbf{H}_k$
- Bias measurement matrix: $\mathbf{G}_k$
- Measurement bias: $\mathbf{b}_k$
- Measurement noise[2]: $\mathbf{w}_k$

Sensor measurements are taken at discrete times and are a linear function of the target state, measurement bias, and measurement noise

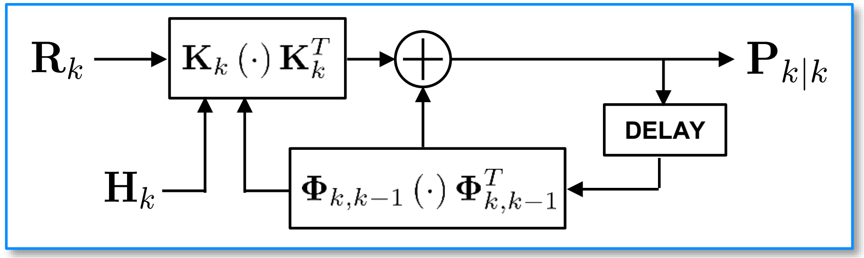
[2] $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$

State Estimation and Error

- For linear system with Gaussian inputs, optimal MMSE estimator is the *Kalman filter*
- If bias is assumed to be negligible (i.e., bias-naïve), the Kalman filter follows:



Bias-Naïve Kalman Filter State



Bias-Naïve Kalman Filter Covariance[1]

- Subsequent focus is the calculation of the estimation error moments for the bias-naïve Kalman filter in the presence of bias

$$\mathbf{m}_k = E[\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k]$$

$$\mathbf{C}_k = E\left[\left(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k - \mathbf{m}_k\right)\left(\hat{\mathbf{x}}_{k|k} - \mathbf{x}_k - \mathbf{m}_k\right)^T\right]$$

Estimation Error Mean and Covariance

[1]Note that state estimate covariance unaffected by presence of bias

Deterministic Bias in a Linear System

- For a deterministic bias trajectory, b_k, the estimation error mean and covariance for a linear system can be written as

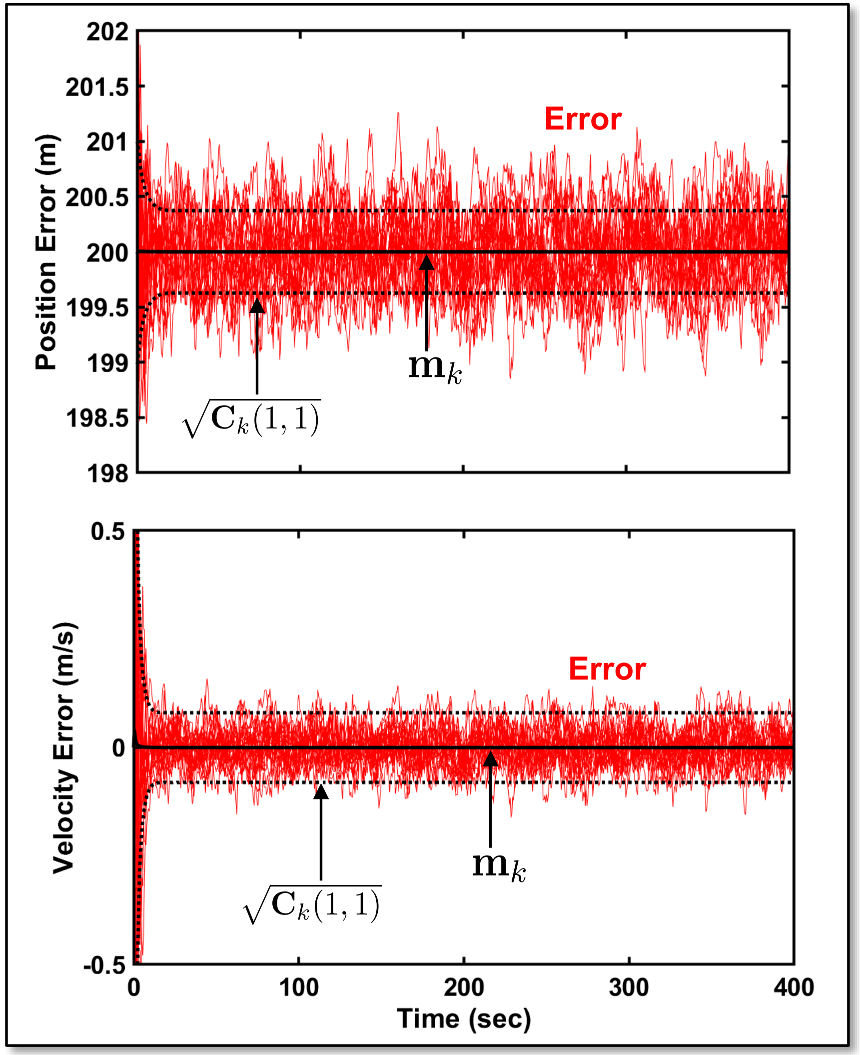
$$\mathbf{m}_k = \sum_{i=1}^{k} \mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i \mathbf{b}_i \qquad \mathbf{C}_k = \mathbf{P}_{k|k}$$

where

$$\mathbf{\Lambda}_i^k = \begin{cases} \prod_{i=1}^{k} (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\, \mathbf{\Phi}_{k,k-1}, & i < k \\ \mathbf{I}, & i = k \end{cases}$$

- $\mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i$ is a weighted projection of the bias at time i into the expected estimation error at time k
- Example—constant velocity target, Cartesian position measurements, and a constant bias

$$\mathbf{R}_k = \mathrm{diag}\left((1\ \mathrm{m})^2, (100\ \mathrm{m})^2, (100\ \mathrm{m})^2 \right) \qquad \mathbf{b}_k = [200\ \mathrm{m}, 50\ \mathrm{m}, -50\ \mathrm{m}]^T$$



Monte Carlo Trials of Example Linear System[1]

[1]20 Monte Carlo trials over measurement noise; Error in x dimension shown

Stochastic Bias in a Linear System

- For a stochastic bias with constant mean, u, and autocovariance, V_k, the estimation error mean and covariance for a linear system can be written as

$$\mathbf{m}_k = \mathbf{S}_k \mathbf{u} \qquad \mathbf{C}_k = \mathbf{P}_{k|k} + \mathbf{B}_k$$

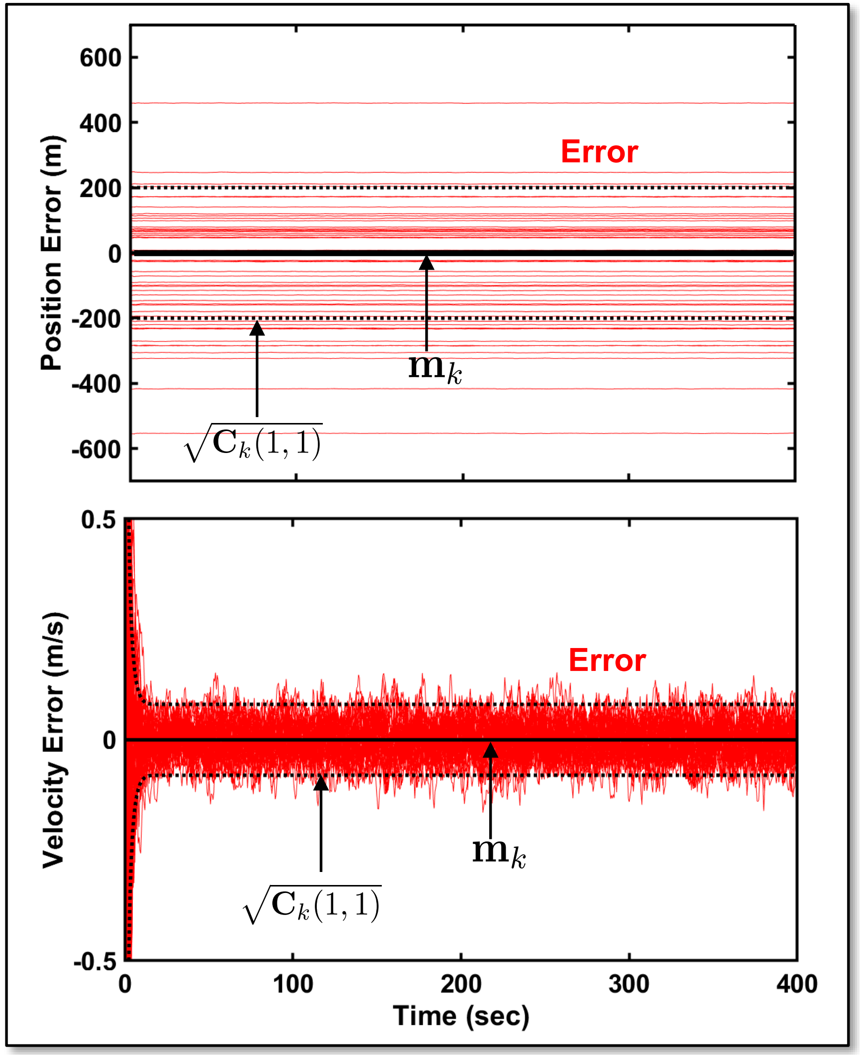
where

$$\mathbf{S}_k = \sum_{i=1}^{k} \mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i$$

$$\mathbf{B}_k = \sum_{i=1}^{k} \sum_{j=1}^{k} \mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i \mathbf{V}_{j-i} \left(\mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i \right)^T$$

- Example (cont'd)—measurement bias is now drawn from a Gaussian distribution

$$E\left[\mathbf{b}\right] = \mathbf{0} \qquad E\left[\mathbf{b}\mathbf{b}^T\right] = \mathrm{diag}\left((200\ \mathrm{m})^2, (50\ \mathrm{m})^2, (50\ \mathrm{m})^2 \right)$$



Monte Carlo Trials of Example Linear System[1]

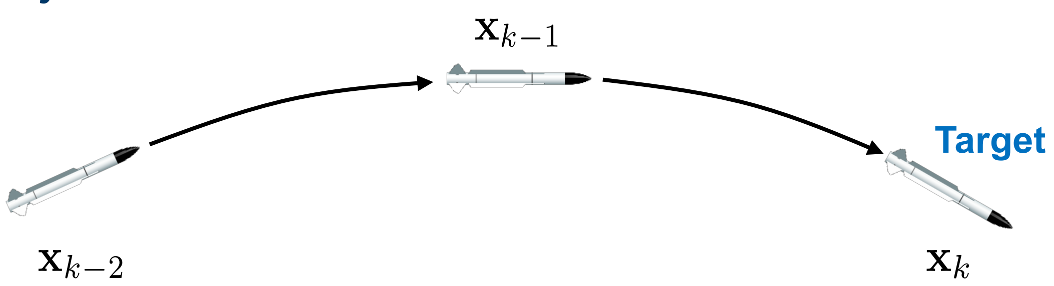
[1]20 Monte Carlo trials over measurement noise and measurement bias; Error in x dimension shown

Outline

- Introduction
- Error Statistics for Linear Systems
- ➡ Error Statistics for Nonlinear Systems
- Bias Significance
- Summary

Nonlinear Systems

Target Dynamics

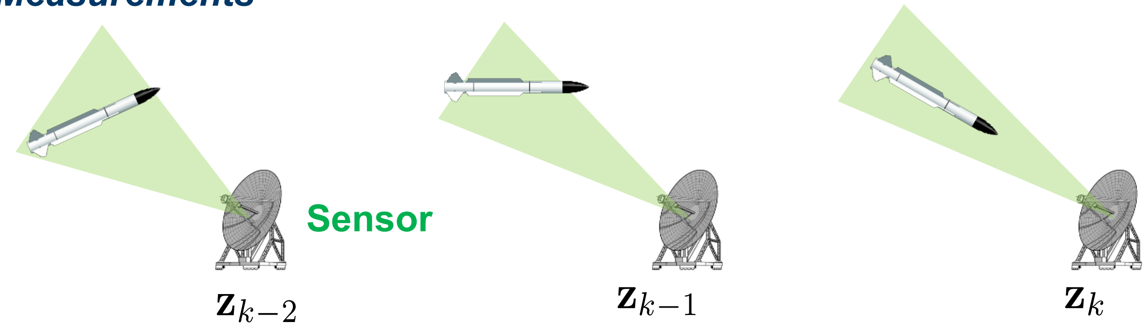


$$\mathbf{x}_k = \phi_{k,k-1}\left(\mathbf{x}_{k-1}\right) + \mathbf{v}_k$$

- Target state → $\mathbf{x}_k$
- State transition function → $\phi_{k,k-1}$
- Process noise[1] → $\mathbf{v}_k$

Target motion described with discrete-time, nonlinear function

Measurements



$$\mathbf{z}_k = h_k\left(\mathbf{x}_k\right) + \mathbf{G}_k\mathbf{b}_k + \mathbf{w}_k$$

- Measurement → $\mathbf{z}_k$
- Measurement function → h_k
- Bias measurement matrix → $\mathbf{G}_k$
- Measurement bias → $\mathbf{b}_k$
- Measurement noise[2] → $\mathbf{w}_k$

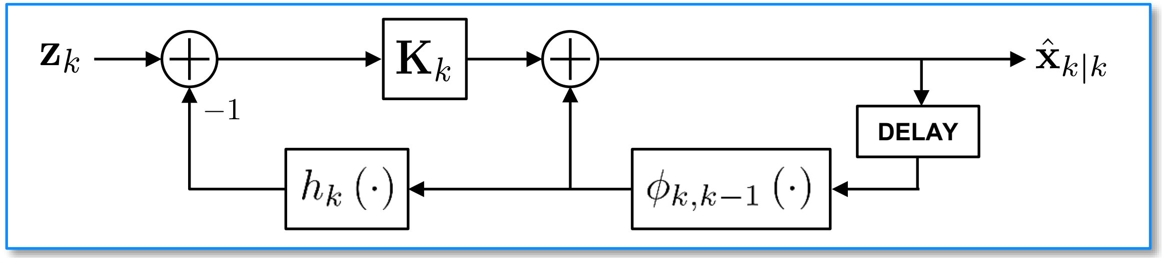
Sensor measurements are taken at discrete times with a nonlinear dependence on the target state

[1] $\mathbf{v}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right)$

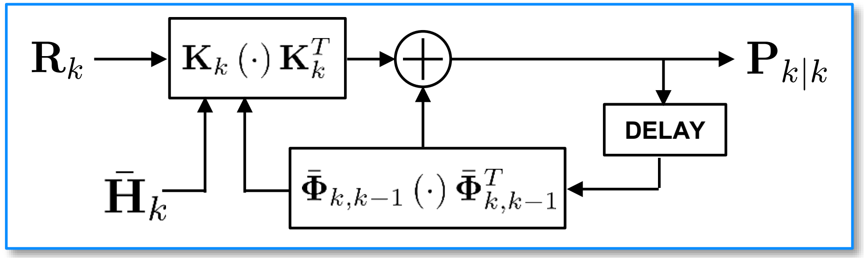
[2] $\mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right)$

Nonlinear Systems and Linearization

- Due to nonlinearities, the Kalman filter is no longer optimal; because of its ubiquity, the extended Kalman filter (EKF) is used for error analysis
- If bias is assumed to be negligible (i.e., bias-naïve), the EKF follows:



Bias-Naïve EKF State



Bias-Naïve EKF Covariance[1]

- Primary approximation of the EKF is the use of Taylor series to represent the nonlinear dynamics and measurement functions

$$\bar{\mathbf{\Phi}}_{k,k-1} = \frac{\partial}{\partial \mathbf{x}} \phi_{k,k-1}(\mathbf{x}) \qquad \bar{\mathbf{H}}_k = \frac{\partial}{\partial \mathbf{x}} h_k(\mathbf{x})$$

Linearized Dynamics and Measurement

  – Linearization point for this work is chosen as the current bias-naïve state

Deterministic Bias in a Nonlinear System

- **For a deterministic bias trajectory, b_k, the estimation error mean and covariance for a linear system can be written as**

$$\mathbf{m}_k = \sum_{i=1}^{k} \mathbf{\Lambda}_i^k \mathbf{K}_i \mathbf{G}_i \mathbf{b}_i \qquad \mathbf{C}_k = \mathbf{P}_{k|k}$$

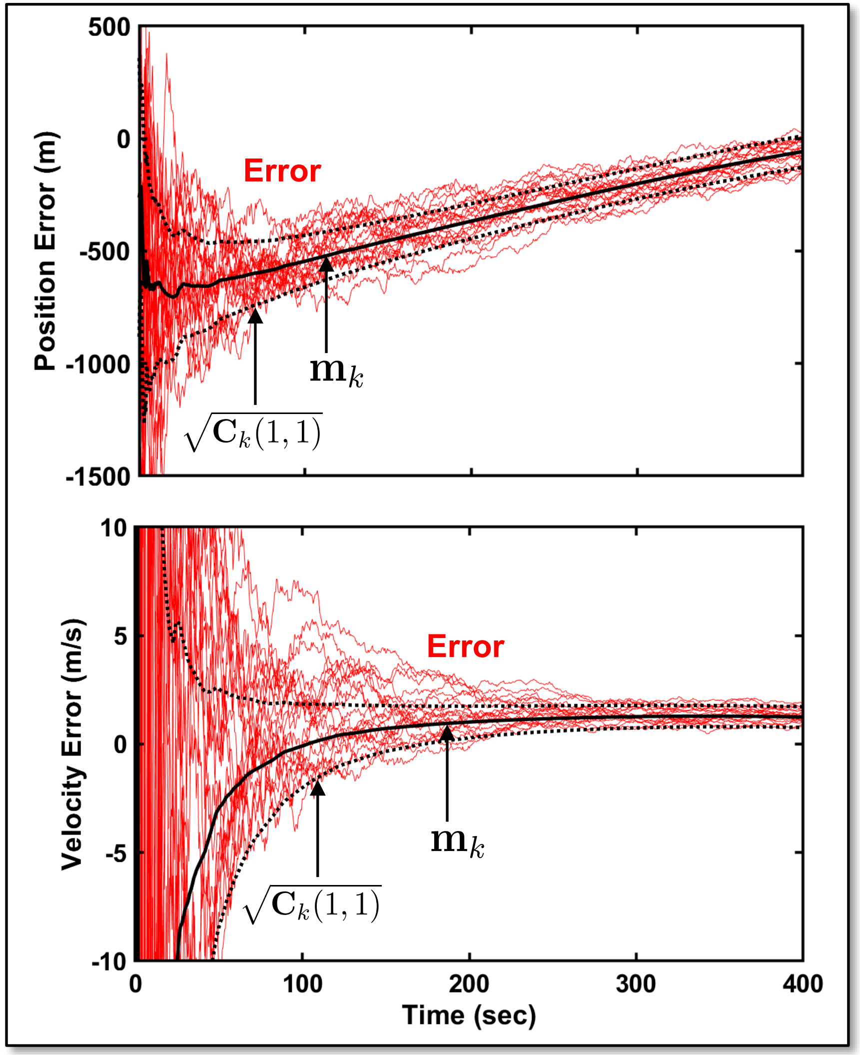
where

$$\mathbf{\Lambda}_i^k = \begin{cases} \prod_{i=1}^{k} \left(\mathbf{I} - \mathbf{K}_k \bar{\mathbf{H}}_k\right) \bar{\mathbf{\Phi}}_{k,k-1}, & i < k \\ \mathbf{I}, & i = k \end{cases}$$

- **Example—ballistic target motion, phased array measurements (RUV), and a constant bias**

$$\mathbf{R}_k = \mathrm{diag}\left((1\ \mathrm{m})^2, (1\ \mathrm{msin})^2, (1\ \mathrm{msin})^2\right) \qquad \mathbf{b}_k = [10\ \mathrm{m}, 0.5\ \mathrm{msin}, -0.5\ \mathrm{msin}]^T$$

- **Main difference is that the expected error is now *time-varying* despite the bias being constant**



Monte Carlo Trials of Example Nonlinear System[1]

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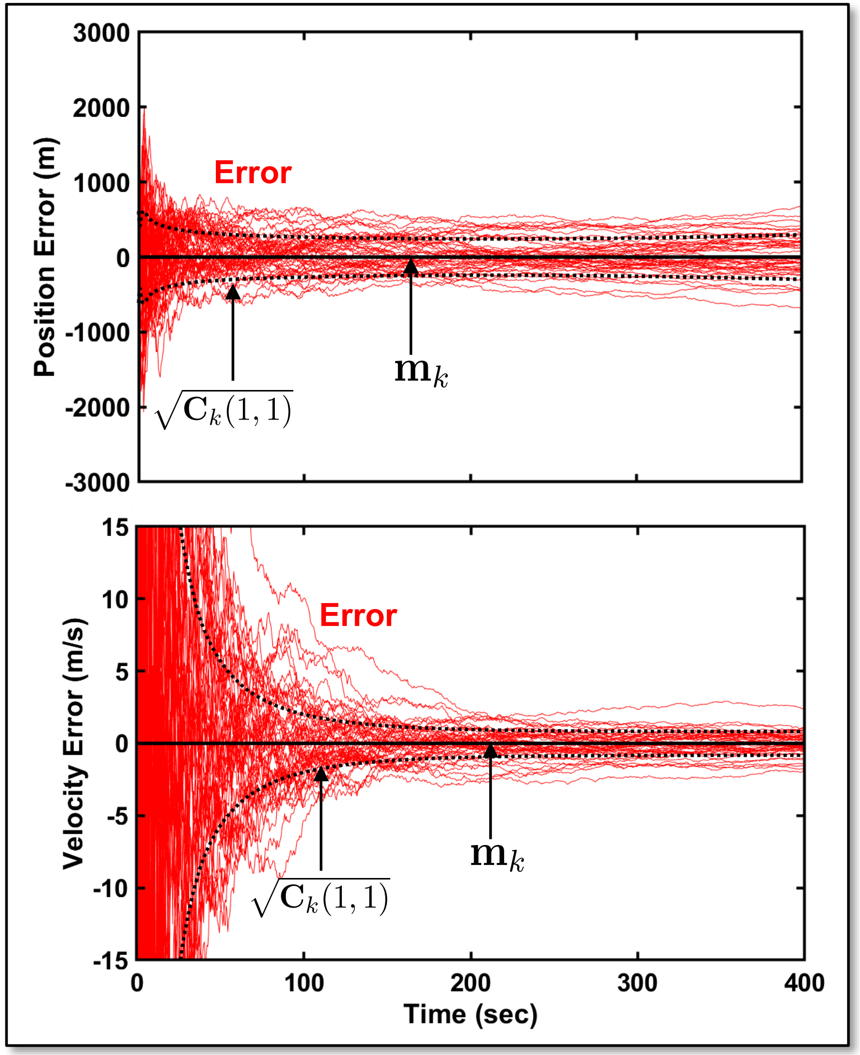
- **Example (cont'd)—measurement bias is now drawn from a Gaussian distribution**

$$E\left[\mathbf{b}\right] = \mathbf{0} \qquad E\left[\mathbf{b}\mathbf{b}^T\right] = \mathrm{diag}\left((10\ \mathrm{m})^2, (0.25\ \mathrm{msin})^2, (0.25\ \mathrm{msin})^2 \right)$$



Monte Carlo Trials of Example Nonlinear System[1]

[1]20 Monte Carlo trials over measurement noise and measurement bias; Error in x dimension shown

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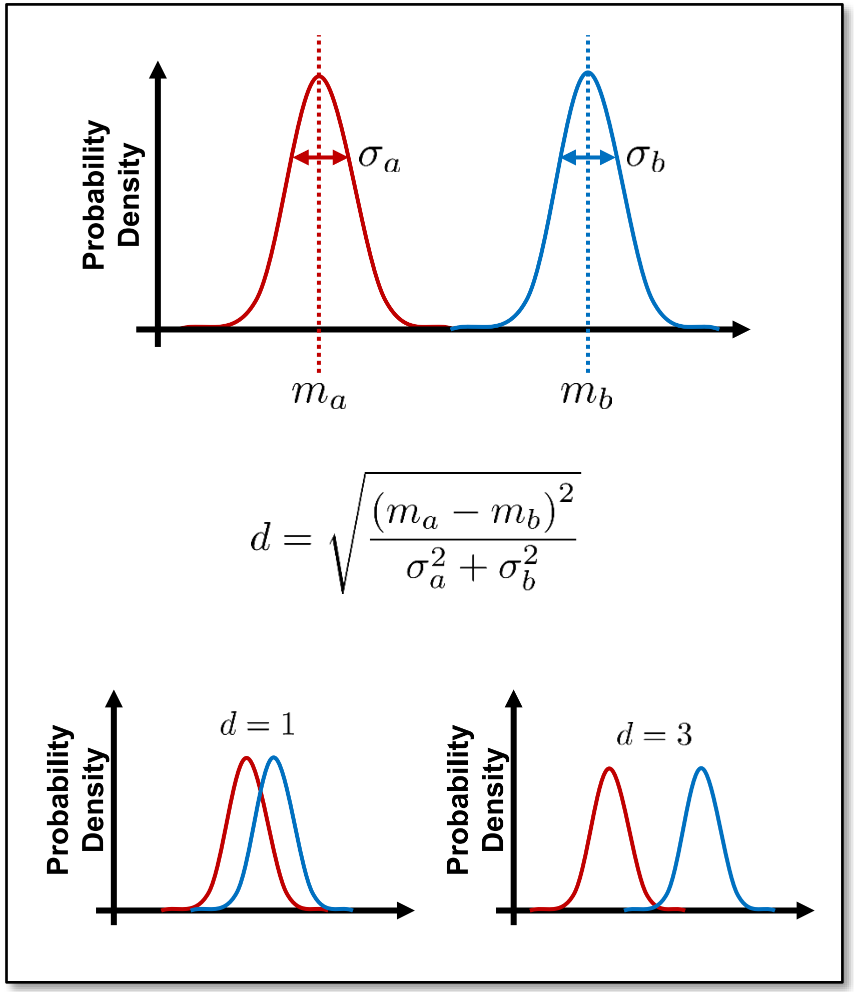
Bias Significance

- **Important to quantify the mismatch in expected and actual error distribution caused by measurement bias**
- **As mismatch increases, performance of track-to-track correlation and multisensor fusion are expected to decrease**
- **Common distance metric for distributions is Mahalanobis distance:**

$$d = \sqrt{(\mathbf{m}_a - \mathbf{m}_b)^T (\mathbf{C}_a + \mathbf{C}_b)^{-1} (\mathbf{m}_a - \mathbf{m}_b)}$$

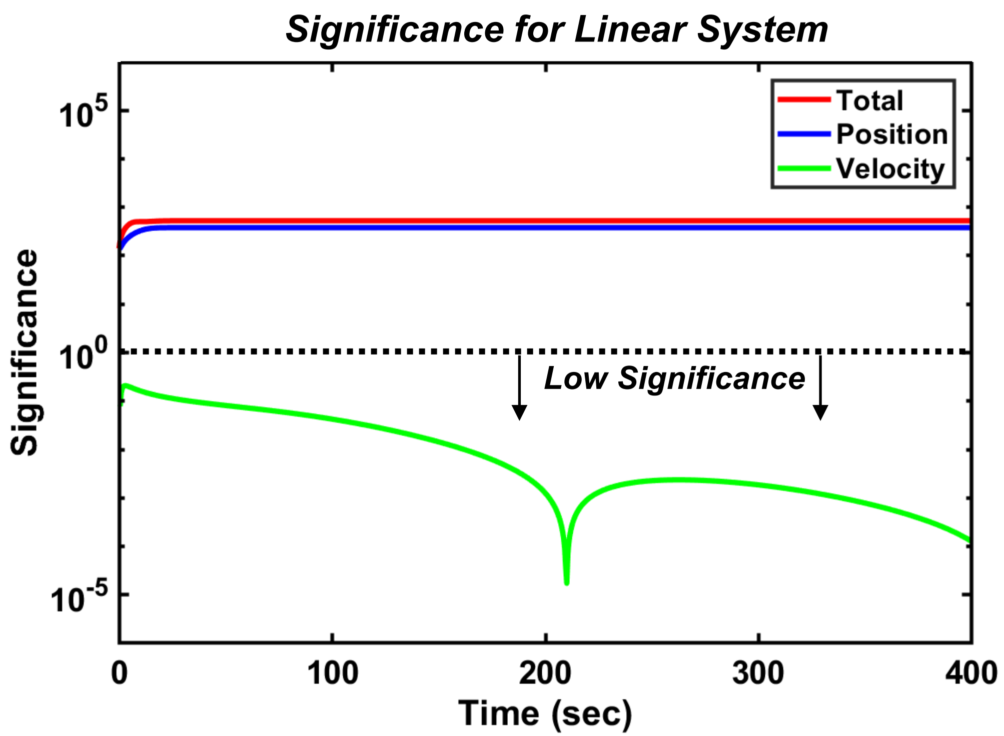
- ***Bias significance,* λ_k, is the Mahalanobis distance between expected and actual error distribution for bias-naïve estimation:**

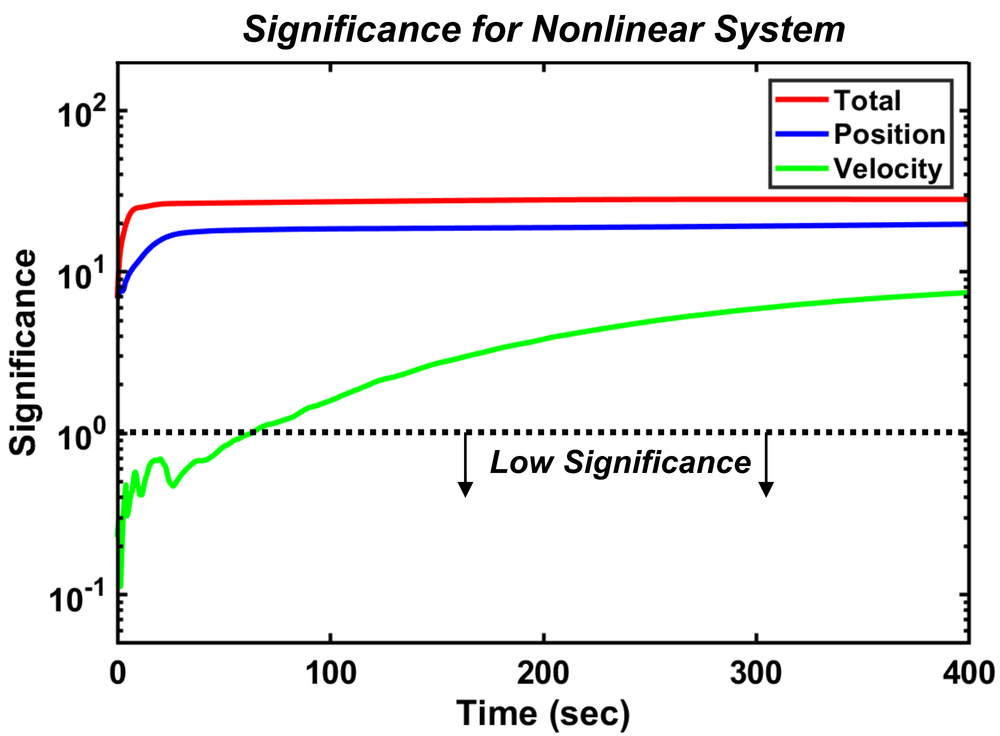
$$\lambda_k = \frac{1}{2}\sqrt{\mathbf{m}_k^T \mathbf{P}_{k|k}^{-1} \mathbf{m}_k}$$



Mahalanobis Distance for Scalar Distributions

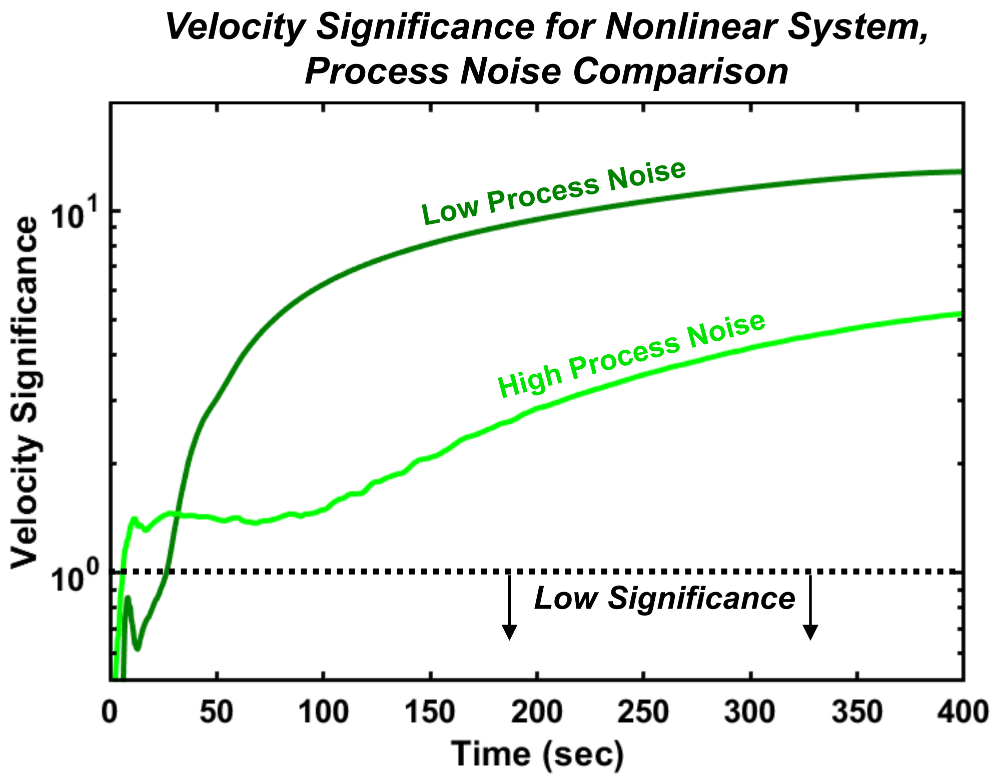
Bias Significance (cont'd)

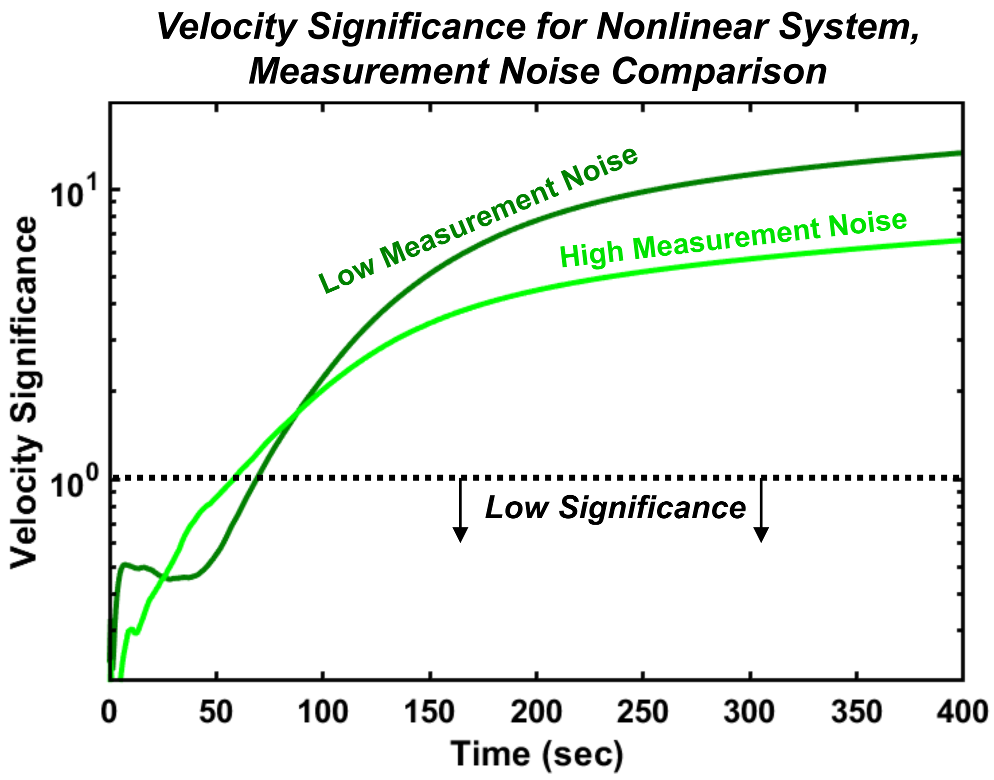




Measurement bias is a clear contributor to position error; nonlinear systems also exhibit appreciable significance in velocity

Measurement and Process Noise Effects





Significance depends on the relative contribution from the multiple error sources; high process and measurement noise can lower significance

Summary

- **Accounting for error sources in estimation is critical for association and fusion of tracking data from multiple sensors**
- **Common nuisance in practical applications is measurement bias, which differs from sensor to sensor**
- **This work analyzes the resultant effects on the estimation error distribution caused by measurement bias for:**
  - **Linear and nonlinear systems**
  - **Deterministic and stochastic biases**
- **Additionally, a quantitative measure of necessity to account for measurement bias is also proposed**
- **Future work includes asymptotic analyses and study of linearization error**